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CRITICAL AND STABLE OUTER-CONNECTED DOMINATION NUMBER

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Abstract

For a given graph G = (V, E), a set D ⊆ V (G) is said to be an outer-connected dominating set if D is dominating and the graph G − D is connected. The outer-connected domination number of a graph G, denoted by $\tilde{\gamma}_c$, is the cardinality of a minimum outer-connected dominating set of G. In this paper we investigate the effects of a vertex removal on the outer-connected domination number of a graph.

Keywords: Domination number, outer-connected domination number.

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1. Introduction

The outer-connected domination was introduced by Cyman in his paper "The outer-connected domination number of a graph" [2]. The study of analysing the effects of removal of a vertex on any domination parameter has remarkable applications in the field of network theory. So, in this paper the effect of a vertex removal on the outer-connected domination number of a graph is being studied.

Let G = (V, E) be a simple graph. The open neighbourhood of a vertex v, denoted by N (v), is the set of all vertices adjacent to v in G and the closed neighbourhood is $N[v] = N(v) \cup \{v\}$. A vertex u is said to be a private neighbour of a vertex v with respect to a set D if $N[u] \cap D = \{v\}$. The private neighbour set of a vertex v with respect to the set D is denoted by $pn[v, D]$. The degree $d_G(v)$, of a vertex v is the number of edges incident to v in G. The minimum and maximum degree among all vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex v of degree $\Delta(G)$ is called a universal vertex, and a vertex of degree one is called a pendant vertex. An edge e with end vertices u and v is denoted by $e = (u, v)$. If u is a pendant vertex, then (u, v) is called a pendant edge. A vertex v of G is called a support if it is adjacent to a pendant vertex. Let Ω be the set of all pendant vertices of G. Let K_n, C_n and P_n denote the complete graph, the cycle and the path of order n, respectively. For positive integers $n_1, n_2, \ldots, n_t$, let $K_{n_1, n_2, \ldots, n_t}$ be the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \cdots \cup S_t$, where $|S_i| = n_i$ for $1 \le i \le t$. A wheel W_n, where $n \ge 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. A subdivision of an edge uv is obtained by removing edge uv, adding a new vertex w, and adding edges uw and vw. A wounded spider is the graph formed by subdividing at most $t-1$ edges of a star $K_{1,t}$. A caterpillar is a tree of order three or more, the removal of whose pendant vertices produces a path. For graph theoretic terminologies which are not specified here, we refer to the book by Chartrand and Lesniak [1].

A set D of vertices of a graph G is said to be a dominating set if every vertex in $V-D$ is adjacent to a vertex in D. A set $D \subseteq V(G)$ is said to be an outer-connected dominating set of G if D is dominating and either $D = V(G)$ or $G-D$ is connected. The cardinality of a minimum outer-connected dominating set in G is called the outer-connected domination number of G and is denoted by $\tilde{\gamma}_c(G)$. An outer-connected dominating set of cardinality $\tilde{\gamma}_c$ is called a $\tilde{\gamma}_c-$ set. For other concepts in connected domination, refer to [3], [4] and [5].

2. Definitions and Preliminary results

Definition 2.1. The vertex set $V(G)$ of a graph G can be partitioned into three sets $\tilde{V}_c^-$, $\tilde{V}_c^+$ and $\tilde{V}_c^0$, according to how the removal of a vertex affects the outer-connected domination number of G. Here,

$$\tilde{V}_c^- = \{v \in V(G) / \tilde{\gamma}_c(G-v) < \tilde{\gamma}_c(G)\}$$

$$\tilde{V}_c^+ = \{v \in V(G) / \tilde{\gamma}_c(G-v) > \tilde{\gamma}_c(G)\} \text{ and}$$

$$\tilde{V}_c^0 = \{v \in V(G) / \tilde{\gamma}_c(G-v) = \tilde{\gamma}_c(G)\}.$$

Example 2.2. Consider the graph G given in Figure 2.1. Here $\tilde{V}_c^- = \{v_5, v_6\}$, $\tilde{V}_c^+ = \{v_1, v_3\}$ and $\tilde{V}_c^0 = \{v_2, v_4\}$.

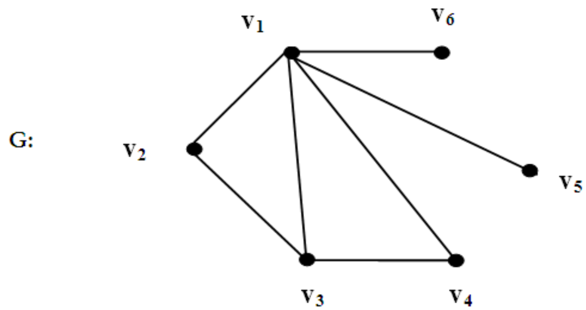

Figure 2.1.

Theorem 2.3. [2]

(i) $\tilde{\gamma}_c(K_n) = 1$ for $n \geq 1$.

(ii) $\tilde{\gamma}_c(\mathrm{C}_n) = n-2$ for $n \geq 3$.

(iii) $$\tilde{\gamma}_c(\mathrm{P}_n) = \begin{cases} n-1, & n = 2,3 \\ n-2, & n \geq 4 \end{cases}$$

(iv) If $t \geq 2$ and $n_1 \leq n_2 \leq \ldots \leq n_t$ then

$$\tilde{\gamma}_c(\mathrm{K}_{n_1, n_2, \ldots, n_t}) = \begin{cases} n_2 & \text{if} \quad t = 2 \text{ and } n_1 = 1, \\ 1 & \text{if} \quad t \geq 3 \text{ and } n_1 = 1, \\ 2 & \text{if} \quad t \geq 2 \text{ and } n_1 > 1. \end{cases}$$

Theorem 2.4. [2] *If G is a connected graph on $n \geq 2$ vertices, then $\tilde{\gamma}_c(G) = n-1$ if and only if G is a star.*

Theorem 2.5. [2] *If $G_1, \ldots, G_r$ are the components of a graph G, then* $\tilde{\gamma}_c(G) = |\mathrm{V(G)}| - \max\{|\mathrm{V}(\mathrm{G}_i)| - \tilde{\gamma}_c(\mathrm{G}_i) : \mathrm{i} = 1, \ldots, \mathrm{r}\}$.

3. Generalized graphs

Theorem 3.1. *For a complete graph on* n *vertices,* $V(K_n)=\tilde{V}_c^0(K_n)$, $n\geq 2.$

Proof. Let v be any vertex of K_n. Then by Theorem 2.3(i), $\tilde{\gamma}_c(K_n-v)=\tilde{\gamma}_c(K_{n-1})=1=\tilde{\gamma}_c(K_n)$, $\forall\, v\in V(K_n)$. Hence, $V(K_n)=\tilde{V}_c^0(K_n)$.

Theorem 3.2. *For a path on* n *vertices,* $V(P_n)=\tilde{V}_c^-(P_n)$, *when* $n\geq 8.$

Proof. Let v be any vertex of P_n. If P_n-v is connected then by Theorem 2.3(iii) $\tilde{\gamma}_c(P_n-v)=n-1-2=n-3<\tilde{\gamma}_c(P_n)=n-2.$ (Here $n-2\geq 4$ as $n\geq 8$). Suppose P_n-v is disconnected. Let P_{m_1}, P_{m_2} be the two components of P_n-v so that $m_1+m_2+1=n.$ Without loss of generality, we assume that $m_1\geq m_2$. By Theorem 2.5, $\tilde{\gamma}_c(P_n-v)=|V(P_n-v)|-max\{|V(P_{m_1})|-\tilde{\gamma}_c(P_{m_1}),$
$|V(P_{m_2})|-\tilde{\gamma}_c(P_{m_2})\}.$

Case 1: Suppose $m_2\leq 3.$ Then $m_1\geq 4.$ By Theorem 2.3(iii), $\tilde{\gamma}_c(P_{m_1})=m_1-2$ and $\tilde{\gamma}_c(P_{m_2})=m_2-1.$ Therefore
$\tilde{\gamma}_c(P_n-v)=n-1-max\{m_1-(m_1-2),\, m_2-(m_2-1)\}=$
$n-1-max\{2,1\}=n-1-2=n-3.$

Case 2: Suppose $m_2\geq 4.$ Then $m_1\geq 4.$ (Since $m_1\geq m_2$). Once again by Theorem 2.3(iii), $\tilde{\gamma}_c(P_{m_1})=m_1-2$ and $\tilde{\gamma}_c(P_{m_2})=m_2-2.$

Therefore

$\tilde{\gamma}_c(Pn-v) = n-1-max\{m_1-(m_1-2), m_2-(m_2-2)\}$
$= n-1-max\{2,2\} = n-3.$

Hence in both the above two case, $\tilde{\gamma}_c(P_n - v) = n-3 < \tilde{\gamma}_c(P_n) = n-2.$ Therefore $V(P_n) = \tilde{V}_c^-(P_n),\ n \geq 8.$

Theorem 3.3. *For a cycle on n vertices,*

$$V(\mathrm{C}_n) = \begin{cases} \tilde{V}_c^0, & n = 3\ or\ 4, \\ \tilde{V}_c^-, & otherwise. \end{cases}$$

Proof. **Case 1 :** Let n = 3. By Theorem 2.3(ii), $\tilde{\gamma}_c(C_3) = 1.$ Also $C_3 - v = P_2$, for any $\mathrm{v} \in \mathrm{V(C_3)}$ and again by Theorem 2.3(iii), $\tilde{\gamma}_c(P_2) = 1.$ Therefore $V(C_3) = \tilde{V}_c^0.$ Now let n = 4. Then by the similar argument $\tilde{\gamma}_c(C_4 - v) = \tilde{\gamma}_c(P_3)$ =2, $\forall\ v \in V(C_4).$ Thus $V(C_4) = \tilde{V}_c^0.$

***Case* 2 :** Let $n \geq 5.$ By Theorem 2.3(*ii*), $\tilde{\gamma}_c(C_n) = n-2$ and $\tilde{\gamma}_c(C_n - v) = \tilde{\gamma}_c(P_{n-1}) = n-3.$ (Since $n-1 \geq 4$). Therefore $\tilde{\gamma}_c(\mathrm{C}_n - \mathrm{v}) < \tilde{\gamma}_c(\mathrm{C}_n),\ \forall\ v \in V(C_n)$ and hence $V(\mathrm{C}_n) = \tilde{\mathrm{V}}_c^-(\mathrm{C}_n).$

Note that if $G = K_2$, then by Theorem 3.1, $V\left(K_2\right) = \tilde{V}_c^0\left(K_2\right).$ Now, in the following theorem we consider a complete bipartite graph other than K_2.

Theorem 3.4. *Let $G = G(V, E)$ be a complete bipartite graph with bipartition $V = V_1 \cup V_2$, where $|V_1| = n_1$ and $|V_2| = n_2$ $(n_2 \geq n_1)$.*

(i) If $n_1 = 1,$ then $v \in \begin{cases} \tilde{V}_c^0(G), & if\ v \in V_1 \\ \tilde{V}_c^-(G), & otherwise. \end{cases}$

(ii)

$$\text{If } n_1 = 2 \text{ and } n_2 > n_1 \text{ then } v \in \begin{cases} \tilde{V}_c^{+}(G), & \text{if } v \in V_1 \\ \tilde{V}_c^{0}(G), & \text{otherwise.} \end{cases}$$

(iii)

If $n_1=n_2=2$ *or* $n_2 \geq n_1 \geq 3$, *then* $V(G) = \tilde{V}_c^0(G)$.

Proof. (i) Suppose $n_1 = 1$. By Theorem 2.3(*iv*), $\tilde{\gamma}_c(G) = n_2$. Let $v \in V_1$. Then $<G-v>$ is totally disconnected. Therefore $\tilde{\gamma}_c(G-v) = n_2 = \tilde{\gamma}_c(G)$ and so $v \in \tilde{V}_c^0(G)$. Suppose $v \notin V_1$. Then $<G-v>$ is again a star graph K_{1,n_2-1}. Then by Theorem 2.4, we have $\tilde{\gamma}_c(G-v) = \tilde{\gamma}_c(K_{1,n_2-1}) = (1+n_2-1)-1 = n_2-1$ $< \tilde{\gamma}_c(G) = n_2$. Hence $v \in \tilde{V}_c^-(G)$.

(ii) Now by Theorem 2.3(*iv*), $\tilde{\gamma}_c(G) = 2$. Let $v \in V_1$. Then $<G-v>$ is a star graph K_{1,n_2}. By Theorem 2.4, $\tilde{\gamma}_c(G-v) = 1 + n_2 - 1 = n_2 > \tilde{\gamma}_c(G) = 2$. Hence $v \in \tilde{V}_c^+(G)$. Suppose $v \notin V_1$. Then $<G-v>$ is again a complete bipartite graph K_{2,n_2-1}. Then by Theorem 2.3(*iv*), $\tilde{\gamma}_c(G-v) = 2 = \tilde{\gamma}_c(G)$. Hence $v \in \tilde{V}_c^0$(G).

(ii) Suppose $n_1 = n_2 = 2$. Then for any vertex $v \in V(G)$, $\tilde{\gamma}_c(G-v) = \tilde{\gamma}_c(P_3) = 2 = \tilde{\gamma}_c(C_4) = \tilde{\gamma}_c(G)$ (by Theorem 2.3(*ii*), (*iii*)). Hence $V(G) = \tilde{V}_c^0(G)$. Suppose $n_2 \geq n_1 \geq 3$. Let v be any vertex of G. Then $<G-v>$ is again a complete bipartite graph. Then by Theorem 2.3(*iv*),

$\tilde{\gamma}_c(G-v)=2=\tilde{\gamma}_c(G)$. Hence $V(G)=\tilde{V}_c^0(G)$.

Theorem 3.5. *Let W_n be a wheel of order n. Then for any vertex $v\in V(W_n)$, we have,*

$$v\in\begin{cases}\tilde{V}_c^0, & if \quad n=4 \; or \; v \; is \; a \; non-universal \; vertex \\ \tilde{V}_c^+, & otherwise\end{cases}.$$

Proof. Let v be the universal vertex of W_n. Then clearly, $\{v\}$ will form a dominating set of W_n and $W_n-v=C_{n-1}$, which is a connected graph. Therefore $\{v\}$ is an outer-connected dominating set of W_n and so $\tilde{\gamma}_c\left(W_n\right)=1$.

Now let $n=4$ and u be any vertex of W_4. Then $\tilde{\gamma}_c(W_4-u)=\tilde{\gamma}_c(C_3)=1=\tilde{\gamma}_c(W_4)$. Hence $u\in\tilde{V}_c^0$.

Now let us assume that $n\geq 5$ and u be any vertex of W_n. Then we have the following cases.

***Case* 1 :** Suppose $u=v$. Then $<W_n-u>$ is a cycle of order $n-1$. Therefore $\tilde{\gamma}_c(W_n-u)=\tilde{\gamma}_c(C_{n-1})=n-1-2=n-3>\tilde{\gamma}_c(W_n)$, ($n-3>1$ as $n\geq 5$). Hence $u\in\tilde{V}_c^+$.

***Case* 2 :** Suppose $u\neq v$. Then the universal vertex $\{u\}$ will form an outer-connected dominating set of W_n-u. Therefore $\tilde{\gamma}_c(W_n-u)=1=\tilde{\gamma}_c(W_n)$ and so $u\in\tilde{V}_c^0(G)$.

4. More Results

Theorem 4.1. *Let p be a pendant vertex of a graph G. Then there exists a minimum outer-connected dominating set D of G such that $p \notin D$ if and only if G is a star.*

Proof. First, let us assume that there exists a minimum outer-connected dominating set D such that $p \notin D$. Since $V(G) - D$ is connected and p is an isolated vertex in $< V(G) - D >$, p must be the only vertex in $V(G) - D$. Therefore $\tilde{\gamma}_c(G) = n - 1$. By Theorem 2.4, G is a star. Converse is obvious.

Observation 4.2. *From the above theorem we can observe that for every graph, other than star, all pendant vertices belong to every outer-connected dominating set.*

Theorem 4.3. *Let $G(\neq K_{1,n}, n \geq 1)$ be a graph and (p, q) be a pendant edge of G. Then for any $\tilde{\gamma}_c$ - set D of G, we have,*

(i) If $q \in D$, then $p \in \tilde{V}_c^-(G)$.

(ii) Let $q \notin D$.

(a) If $q \notin pn[p, D]$, then $p \in \tilde{V}_c^-(G)$.

(b) If $q \in pn[p, D]$, then

$$p \in \begin{cases} \tilde{V}_c^0(G) \cup \tilde{V}_c^-(G), & \text{if } q \text{ is not a cutvertex of } < G - D > \\ \tilde{V}_c^0(G) \cup \tilde{V}_c^+(G), & \text{otherwise.} \end{cases}$$

Proof. (i) Since the only neighbour of p is in D, $D - p$ is a dominating set for $G - p$ and $< (G - p) - (D - p) >$ is connected.

Therefore $D-p$ is an outer-connected dominating set for $G-p$. Thus $\tilde{\gamma}_c(G-p) \le |D-p| < |D| = \tilde{\gamma}_c(G)$. Hence $p \in \tilde{V}_c^-(G)$.

(ii) Let $q \notin D$.

(a) Given $q \notin pn[p,D]$. Then $(D'=)D-p$ will be a dominating set for $G-p$. Also p is not an internal vertex of a path between two vertices in $<(G-p)-D'>$. Therefore $<(G-p)-D'>$ is connected. Hence D' is a an outer-connected dominating set for $G-p$. Therefore $\tilde{\gamma}_c(G-p) \le |D'| < |D| = \tilde{\gamma}_c(G)$ and so $p \in \tilde{V}_c^-(G)$.

(b) Let $q \in pn[p,D]$. Then no vertex of D' will dominate q in $G-p$. Therefore either the vertex q or some vertices have to be selected together with the set D' to form a dominating set for $G-p$. Now we have the following cases.

***Case* 1 :** If q is not a cut vertex of $< G-D >$, then $<(G-p)-(D' \cup \{q\})>$ will be connected and therefore $D' \cup \{q\}$ is an outer-connected dominating set for $G-p$. Thus $\tilde{\gamma}_c(G-p) \le |D' \cup \{q\}| \le |D| = \tilde{\gamma}_c(G)$. Hence $p \in \tilde{V}_c^0(G) \cup \tilde{V}_c^-(G)$.

***Case* 2 :** Now let us assume that q is a cut vertex of $<G-D>$. Then

$<(G-p)-(D'\cup\{q\})>$ is disconnected. Now let C_t be a component of minimum cardinality in $<(G-p)-(D'\cup\{q\})>$. Also the set $D'\cup C_t$ will be a minimum outer-connected dominating set for $G-p$, as the vertices of C_t will dominate the vertex and C_t is minimum. Thus $\tilde{\gamma}_c(G-p)=|D'|+|C_t|\geq |D|=\tilde{\gamma}_c(G)$. Thus $p\in\tilde{V}_c^0(G)\cup\tilde{V}_c^+(G)$.

Theorem 4.4. *Let G be a graph and D be any minimum outer-connected dominating set of G. For every $v\in D$, if $pn[v,D]=\varphi$, then $v\in\tilde{V}_c^-(G)$.*

Proof. Suppose that $pn[v,D]=\varphi$. Then every neighbour of v is adjacent to some vertices of D. Thus $D-v$ is a dominating set for $G-v$. Since $v\in D$ and $<G-D>$ is connected, $<(G-v)-(D-v)>$ is connected. Hence $D-v$ is an outer-connected dominating set of $G-v$. Thus $\tilde{\gamma}_c(\mathrm{G}-\mathrm{v})\leq|\mathrm{D}|-1<|\mathrm{D}|=\tilde{\gamma}_c(G)$. Therefore $v\in\tilde{V}_c^-(G)$.

Theorem 4.5. *Let G be a wounded spider with n vertices. Then*

$$p\in\begin{cases}\tilde{V}_c^0(G)\cup\tilde{V}_c^+(G), & \textit{if} \quad \deg p=\Delta(G)\\ \tilde{V}_c^0(G)\cup\tilde{V}_c^-(G), & \text{otherwise}.\end{cases}$$

Proof. Let G be a wounded spider by subdividing s edges of a star $K_{1,t}$, where $0\leq s\leq t-1$. Let D be the set of all pendant vertices of G. Then $|D|=t$. It can be easily verified that, D forms a outer-connected dominating set for G. Further if D' is an outer-connected dominating set for G other than D. Clearly $G-D'$ can have at most one pendant vertex, say p_1, and therefore $|D'|\geq t-1$. Further to dominate the vertex p_1 at least one non pendant vertex should be included in D'. Thus $|D'|\geq|D|$ and so D is a minimum outer-connected dominating set for G.

(i) Suppose p is a pendant vertex of G and $(p,q)\in E(G)$. Then $q\notin D$. Suppose q is not a private neighbour of p. Then by Theorem 4.3 (*ii*), $p\in \tilde{V}_c^-(G)$. Suppose q is a private neigbour of p in $<V(G)-D>$. Since G is a wounded spider, $deg_G\ q=2$ is either two or $\Delta(G)$. Suppose $deg_G\ q=2$. Then q is adjacent with p and a vertex of maximum degree. Therefore $deg_{G-D}\ q=1$. Thus q is not a cut vertex in $<G-D>$. By Theorem 4.3(*ii*), $v\in \tilde{V}_c^0(G)\cup \tilde{V}_c^-(G)$.

Suppose $deg_G\ q=\Delta(G)$. Since q is the private neigbour of p, then in the star $K_{1,t}$, $t-1$ edges should have been subdivided. Therefore there are $t-1$ vertices of degree two in G. Choose one such vertex, say w. Now the set $D'=(D-\{p\})\cup\{w\}$ dominates $G-\{p\}$. Since $deg_G\ w=2$, w is adjacent to q and a pendant. Therefore $deg_{<G-D>}\ w=1$ so that $<(G-\{p\})-D'>$ is connected. Thus D' forms an outer-connected dominating set for $G-p$. Hence $\tilde{\gamma}_c(G-\{p\})\leq|D'|=|D|=\tilde{\gamma}_c(G)$. Hence $p\in \tilde{V}_c^0(G)\cup \tilde{V}_c^-(G)$.

(ii) Let $deg\ p = 2$. Since $p \in G - D$, D is a dominating set for $G - p$ also. Further $deg\ p = 1$ in $< G - D >$, removal of p will not affect the connectivity of $G - D$. Thus D is a outer-connected dominating set for $G - p$, and so $\tilde{\gamma}_c(G - p) \leq |D| = \tilde{\gamma}_c(G)$. Hence $v \in \tilde{V}_c^0(G) \cup \tilde{V}_c^-(G)$.

(iii) Now let us assume that $deg\ p = \Delta(G)$. Consider $< G - \{p\} >$. Suppose $deg_G\ p = n - 1$. Then $< G - \{p\} >$ is totally disconnected and G is a star $K_{1,n-1}$. Then by Theorem 3.4, we have $p \in \tilde{V}_c^0(G)$. Suppose $deg\ p < n - 1$. Then $< G - \{p\} >$ is a disconnected graph having s copies of K_2, where $s \geq 1$ and $t - s$ copies of K_1. Then by Theorem 2.5,

$$\tilde{\gamma}_c(G - \{p\}) = n - 1 - max\{1, 0\} = n - 1 - 1 = n - 2$$
$$= (1 + t + s) - 2 = t + s - 1 \geq t = \tilde{\gamma}_c(G) \text{ (since } s \geq 1).$$

Hence $p \in \tilde{V}_c^0(G) \cup \tilde{V}_c^+(G)$, if $deg\ p = \Delta(G)$.

Theorem 4.6. *Let G be a caterpillar with n vertices and Ω be the set of all pendant vertices of G. If Ω forms a dominating set for G then any vertex $v \in V(G) - \Omega$,*

$$v \in \begin{cases} \tilde{V}_c^0(G) \cup \tilde{V}_c^-(G), & if \quad \deg_{<V(G)-\Omega>} v = 1 \\ \tilde{V}_c^+(G), & otherwise. \end{cases}$$

Proof. Since G is a caterpillar, degree of any vertex is either one or two in $< V(G) - \Omega >$. Let $v \in V(G) - \Omega$. Since Ω dominates G and $v \in V(G) - \Omega$, Ω dominates $G - \{v\}$.

***Case* 1 :** Suppose $deg\, v=1$ in $<V(G)-\Omega>$. Since $<V(G)-\Omega>$ is connected and $deg\, v=1$ in $<V(G)-\Omega>$, $<(V(G)-\{v\})-\Omega>$ is connected. Thus Ω is an outer-connected dominating set of $G-\{v\}$ and $\tilde{\gamma}_c(G-\{v\}) \leq |\Omega| = \tilde{\gamma}_c(G)$. Therefore $v \in \tilde{V}_c^0(G) \cup \tilde{V}_c^-(G)$.

***Case* 2 :** Suppose $deg\, v=2$ in $<V(G)-\Omega>$. Since G is a caterpillar, $<G-\Omega>$ is a path. Also since $deg\, v=2$, $<(V(G)-\{v\})-\Omega>$ is disconnected into exactly two components. Let C_1 be the minimum cardinality of those components and consider the set $\Omega' = C_1 \cup \Omega.$. Clearly Ω' dominates $G-\{v\}$ and $<G-\{v\}-\Omega'>$ is connected. Therefore Ω' forms a minimum outer-connected dominating set for $G-\{v\}$ (Since and C_1 is minimum). Thus $\tilde{\gamma}_c(G-\{v\}) = |\Omega'| > |\Omega| = \tilde{\gamma}_c(G)$. Hence $v \in \tilde{V}_c^+(G)$.

Theorem 4.7. *Let G be a caterpillar and Ω be the set of all pendant vertices of G. For any minimum outer-connected dominating set D of G, $D-\Omega \subseteq \tilde{V}_c^0(G) \cup \tilde{V}_c^-(G)$.*

Proof: Let $v \in D-\Omega.$. Suppose v has no private neighbours. Then clearly $D-\{v\}$ dominates $G-\{v\}$. Since $<G-D>$ is connected and $v \in D$, $<(G-\{v\})-(D-\{v\})>$ is connected. Therefore $D-\{v\}$ forms an outer-connected dominating set of $D-\{v\}$. Thus $\tilde{\gamma}_c(G-\{v\}) \leq |D|-1 < |D| = \tilde{\gamma}_c(G)$. Hence $v \in \tilde{V}_c^-(G)$.

Suppose v has a private neighbour, say u, in $<G-D>$. Also since $\Omega \subset D$, u is not a pendant vertex of G. Consider $G-\{v\}$.

Clearly u is not dominated by $D-\{v\}$. Therefore consider the set $D'=(D-\{v\})\cup\{u\}$. Since $u\in V(G)-D$ and G is a caterpillar, u lies in the path. Then $deg_{<G-D>}\, u$ is either one or two.

Suppose $deg\, u=2$ in $<V(G)-D>$. Let u_1 and u_2 be the neighbours of u in $<V(G)-D>$. Then v, u_1 and u_2 are non-pendant vertices of G and hence $G-\Omega$ is not a path, which is a contradiction to the fact G is a caterpillar. Therefore $deg\, u\neq 2$. So $deg\, u=1$ in $<V(G)-D>$. Then clearly $<(G-\{v\})-D'>$ is connected. Thus D' is a outer-connected dominating set for $G-\{v\}$ and therefore $\tilde{\gamma}_c(G-\{v\})\leq |D'|=|D|=\tilde{\gamma}_c(G)$. Hence $v\in \tilde{V}_c^0(G)\cup \tilde{V}_c^-(G)$.

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